Reduction of a Class of Three-Loop Vacuum Diagrams to Tetrahedron Topologies

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Abstract

We obtain finite parts (as well as ε-pole parts) of massive three-loop vacuum diagrams with three-point and/or four-point interaction vertices by reducing them to tetrahedron diagrams with both massive and massless lines, whose finite parts were given analytically in a recent paper by Broadhurst. In the procedure of reduction, the method of integration-by-parts recurrence relations is employed. We use our result to compute the $\overline{\mathrm{MS}}$ effective potential of the massive ϕ^4 theory.

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The calculations of the effective potential for a single-component massive ϕ^4 theory [1] and for a massless $O(N)$ ϕ^4 theory [2,3] were achieved at the three-loop level in four-dimensional spacetime. (In three dimensional spacetime, see the work of Rajantie [4] for a single-component ϕ^4 theory.) The calculations in Ref. 1 and Refs. 2 and 3 are done in the dimensional regularization scheme with a specific set of renormalization conditions. The same calculations at a lower-loop level, in the cutoff regularization, with the same renormalization conditions can be found in Ref. 5 and Ref. 6 respectively. We see that the results agree with each other. Therefore, in the mass-dependent scheme, we do not need to calculate finite the parts of three-loop diagrams. Knowledge of the pole terms is sufficient.

However, in a mass-independent scheme, such as the MS or $\overline{\rm MS}$ scheme, we have to calculate three-loop diagrams to the finite parts, i.e., to the ε^0 order in the ε expansion. Without imposing renormalization conditions at a specific scale, we just leave an arbitrary constant μ, which is introduced inevitably for dimensional reasons, unspecified as in Eq. (13) below. This has the drawback that it does not involve true physical parameters measured at a given scale. Though it normally takes some effort to express physically measurable quantities in terms of the parameters of the expression, the renormalization group (RG) equation is dealt with much easier,[1] and the calculations in complicated theories are much more convenient.

The purpose of this note is to reduce a class of massive three-loop vacuum diagrams to (three-loop) tetrahedron diagrams with both massive and massless lines by using the method of recurrence relations. Once the reductions are completed, the finite parts (as well as the ε-pole parts) of the diagrams in question can be readily obtained because the finite parts of these tetrahedron diagrams were determined analytically by Broadhurst [8].

Let us define three-loop vacuum integrals J, K, and L, which are nonfactorizable into lower-loop integrals:

$$\begin{aligned}
J &\equiv \int_{kpq} \frac{1}{(p^2+\sigma^2)[(p+k)^2+\sigma^2](q^2+\sigma^2)[(q+k)^2+\sigma^2]} , \\
K &\equiv \int_{kpq} \frac{1}{(k^2+\sigma^2)(p^2+\sigma^2)[(p+k)^2+\sigma^2](q^2+\sigma^2)[(q+k)^2+\sigma^2]} , \\
L &\equiv \int_{kpq} \frac{1}{(k^2+\sigma^2)^2(p^2+\sigma^2)[(p+k)^2+\sigma^2](q^2+\sigma^2)[(q+k)^2+\sigma^2]} .
\end{aligned} \tag{1}$$

The momenta in Eq. (1) are all (Wick-rotated) Euclidean, and the abbreviated integration measure is defined as

$$\int_k = \mu^{4-n} \int \frac{d^n k}{(2\pi)^n} , \tag{2}$$

where $n = 4-2\varepsilon$ is the space-time dimension in the framework of dimensional regularization, and μ is an arbitrary constant with a dimension of mass. The pole parts of all the integrals in Eq. (1) are known [1, 3]. Now, the problem is to the find the finite parts of these integrals in terms of the known transcendental numbers.

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[1]See Ref. 7 for two-loop RG improvement of the effective potential.

While massless multi-loop diagrams can be dealt with by essentially algebraic methods [9], the situation is more complicated in the case of *massive* diagrams. In notation of Avdeev [10], the above three-loop diagrams J, K, and L correspond to $B_N(0,0,1,1,1,1)$, $D_5(1,1,1,1,1,0)$, and $D_5(1,1,1,1,2,0)$, respectively. The method of integration-by-parts recurrence relations, given in Avdeev's paper [10], allows us to connect the integrals $B_N(0,0,1,1,1,1)$, $D_5(1,1,1,1,1,0)$, and $D_5(1,1,1,1,2,0)$ to the tetrahedron integrals $B_N(1,1,1,1,1,1)$, $B_M(1,1,1,1,1,1)$, and $D_5(1,1,1,1,1,1)$. The analytic expressions of the finite parts, i.e., ε^0 order terms, for all tetrahedron vacuum diagrams with different combinations of massless and massive lines of a single mass scale were given in a recent paper by Broadhurst [8]. With the convention of integration measure used in Ref. 8,

$$\int [dk] = \int \frac{d^n k}{\sigma^{n-4}\pi^{n/2}\Gamma(1+\varepsilon)},$$

we quote the results of Broadhurst which are relevant to our calculation:

$$\begin{aligned}
B_N(1,1,1,1,1,1) = V_{4N} &= \frac{2\zeta(3)}{\varepsilon} + 6\zeta(3) - 14\zeta(4) - 16U_{3,1}\ , \\
B_M(1,1,1,1,1,1) = V_{3T} &= \frac{2\zeta(3)}{\varepsilon} + 6\zeta(3) - 9\zeta(4)\ , \\
D_5(1,1,1,1,1,1) = V_5 &= \frac{2\zeta(3)}{\varepsilon} + 6\zeta(3) - \frac{469}{27} + \frac{8}{3}\mathrm{Cl}_2^2\Big(\frac{\pi}{3}\Big) - 16V_{3,1}\ ,
\end{aligned} \tag{3}$$

where $U_{3,1}$ and $V_{3,1}$ are defined as

$$\begin{aligned}
U_{3,1} &\equiv \sum_{m>n>0} \frac{(-1)^{m+n}}{m^3 n}\ , \\
V_{3,1} &\equiv \sum_{m>n>0} \frac{(-1)^m \cos(2\pi n/3)}{m^3 n}\ ,
\end{aligned}$$

and can be expressed in terms of known transcendental numbers as[2]

$$\begin{aligned}
U_{3,1} &= \frac{\zeta(4)}{2} + \frac{\zeta(2)}{2}\ln^2 2 - \frac{1}{12}\ln^4 2 - 2\mathrm{Li}_4\Big(\frac{1}{2}\Big)\ , \\
V_{3,1} &= \frac{1}{3}\mathrm{Cl}_2^2\Big(\frac{\pi}{3}\Big) - \frac{1}{4}\pi\mathrm{Ls}_3\Big(\frac{2\pi}{3}\Big) + \frac{13}{24}\zeta(3)\ln 3 - \frac{259}{108}\zeta(4) + \frac{3}{8}\mathrm{Ls}_4^{(1)}\Big(\frac{2\pi}{3}\Big)\ .
\end{aligned} \tag{4}$$

In the above equation, $\mathrm{Li}_4(x)$, $\mathrm{Cl}_2(x)$, $\mathrm{Ls}_3(x)$, and $\mathrm{Ls}_4^{(1)}(x)$ are the polylogarithm, Clausen's polylogarithm, the log-sine integral, and the generalized log-sine integral, respectively [12], whose numerical values at the given arguments are

$$\mathrm{Li}_4\Big(\frac{1}{2}\Big) = 0.517\ 479\ 061...\ ,$$

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[2]The combination $V_{3,1}$ in Eq. (4) in terms of the known transcendental numbers was found by Fleischer and Kalmykov [11].

$$\mathrm{Cl}_2\left(\frac{\pi}{3}\right) = 1.014\ 941\ 606...\ ,$$
$$\mathrm{Ls}_3\left(\frac{2\pi}{3}\right) = -2.144\ 762\ 212...\ ,$$
$$\mathrm{Ls}_4^{(1)}\left(\frac{2\pi}{3}\right) = -0.497\ 675\ 551...\ . \tag{5}$$

Using the the method of recurrence relations [9, 10, 13], we can arrive at following connections:

$$\begin{aligned}
\sigma^{-4}B_N(0,0,1,1,1,1) &= 32B_4\Big[\frac{1}{2(1-3\varepsilon)}+\frac{4}{2-3\varepsilon}-\frac{2}{1-2\varepsilon}-\frac{1}{2(1-\varepsilon)}\Big] \\
&-\frac{486}{1-3\varepsilon}-\frac{729}{2(2-3\varepsilon)}-\frac{35}{2\varepsilon}+\frac{3}{\varepsilon^2}+\frac{2}{\varepsilon^3}+\frac{512}{1-2\varepsilon}+\frac{10}{1-\varepsilon} \\
&+\frac{\Gamma(1-\varepsilon)\Gamma^2(1+2\varepsilon)\Gamma(1+3\varepsilon)}{\Gamma^2(1+\varepsilon)\Gamma(1+4\varepsilon)}\Big[\frac{14}{3\varepsilon^2}+\frac{35}{\varepsilon}+\frac{378}{1-3\varepsilon}+\frac{189}{2-3\varepsilon}-\frac{896}{3(1-2\varepsilon)}-\frac{14}{3(1-\varepsilon)}\Big]\ , \\
D_5(1,1,1,1,1,1) &= -\frac{2}{3}\sigma^{-2}D_5(1,1,1,1,1,0)\left[1-\frac{1}{2\varepsilon}\right] \\
&-\frac{8}{3}B_4\Big[\frac{3}{2(1-3\varepsilon)}-\frac{2}{1-2\varepsilon}+\frac{1}{2(1-\varepsilon)}\Big]-\frac{2}{3\varepsilon^2}\sigma^{-2}\mathbf{VL111}(1,1,1) \\
&-\frac{2}{3\varepsilon^4}-\frac{1}{\varepsilon^3}+\frac{4}{3\varepsilon^2}+\frac{243}{2(1-3\varepsilon)}+\frac{15}{\varepsilon}-\frac{160}{3(1-2\varepsilon)}+\frac{7}{6(1-\varepsilon)} \\
&-\frac{\Gamma(1-\varepsilon)\Gamma^2(1+2\varepsilon)\Gamma(1+3\varepsilon)}{\Gamma^2(1+\varepsilon)\Gamma(1+4\varepsilon)}\Big[\frac{7}{9\varepsilon^3}+\frac{14}{3\varepsilon^2}+\frac{175}{9\varepsilon}+\frac{189}{2(1-3\varepsilon)}-\frac{224}{9(1-2\varepsilon)}+\frac{7}{18(1-\varepsilon)}\Big]\ , \\
D_5(1,1,1,1,2,0) &= \frac{1}{3}\sigma^{-2}D_5(1,1,1,1,1,0)(1+\varepsilon)-\frac{32}{3}B_4\Big[\frac{1}{2(1-3\varepsilon)}-\frac{1}{1-2\varepsilon}+\frac{1}{2(1-\varepsilon)}\Big] \\
&-\frac{4}{3\varepsilon}\sigma^{-2}\mathbf{VL111}(1,1,1)+\frac{162}{1-3\varepsilon}+\frac{44}{3\varepsilon}-\frac{4}{3\varepsilon^3}-\frac{256}{3(1-2\varepsilon)}+\frac{10}{3(1-\varepsilon)} \\
&-\frac{\Gamma(1-\varepsilon)\Gamma^2(1+2\varepsilon)\Gamma(1+3\varepsilon)}{\Gamma^2(1+\varepsilon)\Gamma(1+4\varepsilon)}\Big[\frac{28}{9\varepsilon^2}+\frac{56}{3\varepsilon}+\frac{126}{1-3\varepsilon}-\frac{448}{9(1-2\varepsilon)}+\frac{14}{9(1-\varepsilon)}\Big]\ ,
\end{aligned} \tag{6}$$

where B_4 is proportional to the difference between $B_N(1,1,1,1,1,1)$ and $B_M(1,1,1,1,1,1)$ [13], i.e.,

$$B_4 = -\frac{(1-2\varepsilon)(2-2\varepsilon)}{4}\Big[B_M(1,1,1,1,1,1)-B_N(1,1,1,1,1,1)\Big]\ .$$

In Eq. (6), $\mathbf{VL111}$(1,1,1) denotes a two-loop bubble integral, shown in Fig. 1 of the paper by Fleischer and Kalmykov [11]. Its value up to ε^3 order in the ε expansion can be found in Eq. (7) of Ref. 11. For all higher-order terms, one may refer to Ref. 14. For our desired accuracy, it is sufficient to take its value up to ε order:

$$\begin{aligned}
\sigma^{-2}\mathbf{VL111}(1,1,1) = -\frac{3}{2(1-\varepsilon)(1-2\varepsilon)}\Big[&\frac{1}{\varepsilon^2}-\frac{4}{\sqrt{3}}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) \\
&+\varepsilon\Big\{\frac{4}{\sqrt{3}}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln 3-\frac{\pi^3}{3\sqrt{3}}-2\sqrt{3}\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)\Big\}+O(\varepsilon^2)\Big]\ .
\end{aligned} \tag{7}$$

From Eqs. (3) — (7), we eventually obtain

$$
\begin{aligned}
B_N(0,0,1,1,1,1) &= \frac{2}{\varepsilon^3} + \frac{23}{3\varepsilon^2} + \frac{35}{2\varepsilon} + \frac{275}{12} + O(\varepsilon) \ , \\
D_5(1,1,1,1,1,0) &= -\frac{1}{\varepsilon^3} - \frac{17}{3\varepsilon^2} + \frac{1}{\varepsilon}\Big[-\frac{67}{3} + \frac{12}{\sqrt{3}}\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big)\Big] - \frac{229}{3} + \frac{60}{\sqrt{3}}\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big) \\
&\quad -\frac{12}{\sqrt{3}}\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big)\ln 3 + \frac{\pi^3}{\sqrt{3}} + 6\zeta(3) + 6\sqrt{3}\mathrm{Ls}_3\Big(\frac{2\pi}{3}\Big) + O(\varepsilon) \ , \\
D_5(1,1,1,1,2,0) &= \frac{1}{3\varepsilon^3} + \frac{2}{3\varepsilon^2} + \frac{1}{\varepsilon}\Big[\frac{2}{3} - \frac{4}{\sqrt{3}}\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big)\Big] \\
&\quad -\frac{2}{3} + \frac{4}{\sqrt{3}}\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big)\ln 3 + 2\zeta(3) - \frac{\pi^3}{3\sqrt{3}} - 2\sqrt{3}\mathrm{Ls}_3\Big(\frac{2\pi}{3}\Big) + O(\varepsilon) \ . \qquad (8)
\end{aligned}
$$

We readily recover the values of J, K, and L in our original integration measure, Eq. (2), by using the following relations:

$$
\begin{aligned}
J &= \frac{\sigma^4}{(4\pi)^6}\Big(\frac{\sigma^2}{4\pi\mu^2}\Big)^{-3\varepsilon} \exp\Big[-3\gamma\varepsilon + \frac{3\zeta(2)}{2}\varepsilon^2 - \zeta(3)\varepsilon^3 + \cdots\Big] B_N(0,0,1,1,1,1) \ , \\
K &= \frac{\sigma^2}{(4\pi)^6}\Big(\frac{\sigma^2}{4\pi\mu^2}\Big)^{-3\varepsilon} \exp\Big[-3\gamma\varepsilon + \frac{3\zeta(2)}{2}\varepsilon^2 - \zeta(3)\varepsilon^3 + \cdots\Big] D_5(1,1,1,1,1,0) \ , \\
L &= \frac{1}{(4\pi)^6}\Big(\frac{\sigma^2}{4\pi\mu^2}\Big)^{-3\varepsilon} \exp\Big[-3\gamma\varepsilon + \frac{3\zeta(2)}{2}\varepsilon^2 - \zeta(3)\varepsilon^3 + \cdots\Big] D_5(1,1,1,1,2,0) \ ,
\end{aligned}
$$

where the exp[ ] factor comes from an expansion of $\Gamma^3(1+\varepsilon)$ and γ is the Euler constant.

In the standard $\overline{\mathrm{MS}}$ scheme [15], the factors $\ln(4\pi)$ and γ are absorbed into the renormalization scale μ. Howerever, in the other widespread $\overline{\mathrm{MS}}$ convention (see, e.g., Refs. 16 and 17), the factor $\zeta(2)$ is absorbed further into the scale μ. (This convention gives the same result in the one-loop diagrams and is more convenient in higher-loop massive calculations.) By introducing a new renormalization scale $\bar{\mu}$,

$$
\bar{\mu}^2 = 4\pi\mu^2 \exp\Big[-\gamma + \frac{\zeta(2)\varepsilon}{2}\Big] \ , \qquad (9)
$$

the above three integrals J, K, and L are given as follows:

$$
\begin{aligned}
J &= \frac{\sigma^4}{(4\pi)^6}\Big(\frac{\sigma^2}{\bar{\mu}^2}\Big)^{-3\varepsilon}\Big[\frac{2}{\varepsilon^3} + \frac{23}{3\varepsilon^2} + \frac{35}{2\varepsilon} + F_J\Big] \ , \\
K &= \frac{\sigma^2}{(4\pi)^6}\Big(\frac{\sigma^2}{\bar{\mu}^2}\Big)^{-3\varepsilon}\Big[-\frac{1}{\epsilon^3} - \frac{17}{3\varepsilon^2} + \frac{1}{\varepsilon}\Big\{-\frac{67}{3} + \frac{12}{\sqrt{3}}\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big)\Big\} + F_K\Big] \ , \\
L &= \frac{1}{(4\pi)^6}\Big(\frac{\sigma^2}{\bar{\mu}^2}\Big)^{-3\varepsilon}\Big[\frac{1}{3\varepsilon^3} + \frac{2}{3\varepsilon^2} + \frac{1}{\varepsilon}\Big\{\frac{2}{3} - \frac{4}{\sqrt{3}}\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big)\Big\} + F_L\Big] \ , \qquad (10)
\end{aligned}
$$

where

$$
\begin{aligned}
F_J &= \frac{275}{12} - 2\zeta(3) \ , \\
F_K &= -\frac{229}{3} + \frac{\pi^3}{\sqrt{3}} + 7\zeta(3) + \frac{60}{\sqrt{3}}\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big) - \frac{12}{\sqrt{3}}\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big)\ln 3 + 6\sqrt{3}\mathrm{Ls}_3\Big(\frac{2\pi}{3}\Big) \ , \\
F_L &= -\frac{2}{3} - \frac{\pi^3}{3\sqrt{3}} + \frac{5\zeta(3)}{3} + \frac{4}{\sqrt{3}}\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big)\ln 3 - 2\sqrt{3}\mathrm{Ls}_3\Big(\frac{2\pi}{3}\Big) \ . \qquad (11)
\end{aligned}
$$

This completes the analytic evaluations of the three-loop vacuum integrals of Eq. (1) up to the finite parts.

The purely numerical computation of the finite parts for some three-loop vacuum diagrams in a paper by Pelissetto and Vicari [18] enables us to extract the numerical values for F_J, F_K, and F_L. In obtaining F_J and F_K, we assume first the *unknown* finite parts F_J and F_K for J and K in Eq. (10) since the pole parts are already known [1,3]. Then, we differentiate J and K, twice and once, respectively, with respect to σ^2. Meanwhile, we can differentiate J and K in Eq. (1), twice and once, respectively, with respect to σ^2 before the momentum integrations, yielding the three-loop diagrams given in Appendix B of Ref. 18 whose finite parts have been numerically calculated. By equating the results of the differentiations thus done, we determine the unknown values of F_J and F_K. The extracted results are

$$\begin{aligned}
F_J &= \frac{275}{12} - 6\sqrt{3}\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big) - 22\zeta(3) + 80S_1 + 20S_2 - 120S_5 - 30S_6 - 6S_7 \ , \\
F_K &= -\frac{229}{3} + \frac{28}{\sqrt{3}}\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big) - \pi^2 + 9\zeta(3) - 24S_1 - 6S_2 + 6S_4 - 12S_7 \ , \\
F_L &= -\frac{2}{3} - 4\sqrt{3}\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big) - \frac{\pi^2}{3} - \frac{5\zeta(3)}{3} + 8S_1 + 2S_2 + 2S_4 \ ,
\end{aligned} \tag{12}$$

where the quantities S_1, S_2, S_4, S_5, S_6, and S_7 were calculated numerically in Appendix B of Ref. 18. We see that the numerical values of F_J, F_K, and F_L given in Eq. (12) agree with the analytical values in Eq. (11).[3]

Our results, Eqs. (10) and (11), together with the result for the all-massive-line tetrahedron diagram in Ref. 8,

$$\begin{aligned}
M &\equiv \int_{kpq} \frac{1}{(k^2+\sigma^2)(p^2+\sigma^2)(q^2+\sigma^2)[(k-p)^2+\sigma^2][(p-q)^2+\sigma^2][(q-k)^2+\sigma^2]} \\
&= \frac{1}{(4\pi)^6}\Big(\frac{\sigma^2}{\bar{\mu}^2}\Big)^{-3\varepsilon}\Big[\frac{2\zeta(3)}{\varepsilon} + 6\zeta(3) - 17\zeta(4) - \frac{2\pi^2}{3}\ln^2 2 + \frac{2}{3}\ln^4 2 - 4\mathrm{Cl}_2^2\Big(\frac{\pi}{3}\Big) + 16\mathrm{Li}_4\Big(\frac{1}{2}\Big)\Big] \ ,
\end{aligned}$$

enable us to calculate the three-loop effective potential in the $\overline{\mathrm{MS}}$ scheme for the single-component massive ϕ^4 theory. The renormalization of the three-loop effective potential is straightforward, albeit long. Thus, we simply report the result:

$$\begin{aligned}
V &= V^{(0)} + \hbar V^{(1)} + \hbar^2 V^{(2)} + \hbar^3 V^{(3)} + O(\hbar^4) \ , \\
V^{(0)} &= \frac{m^2\phi^2}{2} + \frac{\lambda\phi^4}{4!} \ , \\
V^{(1)} &= \frac{\lambda}{(4\pi)^2}\Big[-\frac{3m^4}{8\lambda} - \frac{3m^2\phi^2}{8} - \frac{3\lambda\phi^4}{32} + \Big\{\frac{m^4}{4\lambda} + \frac{m^2\phi^2}{4} + \frac{\lambda\phi^4}{16}\Big\}\ln\Big(\frac{m_\phi^2}{\bar{\mu}^2}\Big)\Big] \ ,
\end{aligned}$$

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[3] Also, in a recent paper by Anderson et al. [19], F_J was calculated numerically. The numerical value of C_0 in Eq. (18) of Ref. 19 agrees with our analytic value F_J in Eq. (11): our $F_J + 23\pi^2/12$ means C_0 of Ref. 19.

$$
\begin{aligned}
V^{(2)} &= \frac{\lambda^2}{(4\pi)^4}\Big[\frac{m^4}{8\lambda} + m^2\phi^2\Big(\frac{3}{4} - \frac{1}{2\sqrt{3}}\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big)\Big) + \lambda\phi^4\Big(\frac{11}{32} - \frac{1}{4\sqrt{3}}\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big)\Big) \\
&\quad - \Big\{\frac{m^4}{4\lambda} + \frac{3m^2\phi^2}{4} + \frac{5\lambda\phi^4}{16}\Big\}\ln\Big(\frac{m_\phi^2}{\bar{\mu}^2}\Big) + \Big\{\frac{m^4}{8\lambda} + \frac{m^2\phi^2}{4} + \frac{3\lambda\phi^4}{32}\Big\}\ln^2\Big(\frac{m_\phi^2}{\bar{\mu}^2}\Big)\Big]\,, \\
V^{(3)} &= \frac{\lambda^3}{(4\pi)^6}\Big[\frac{1}{576}\frac{m^4}{\lambda} + m^2\phi^2\Big(-\frac{2363}{576} + \frac{13}{4\sqrt{3}}\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big) + \frac{3\zeta(3)}{4}\Big) \\
&\quad + \lambda\phi^4\Big(-\frac{4487}{2304} + \frac{11}{8\sqrt{3}}\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big) + \frac{1}{6}\mathrm{Cl}_2^2\Big(\frac{\pi}{3}\Big) - \frac{2}{3}\mathrm{Li}_4\Big(\frac{1}{2}\Big) + \frac{17\zeta(4)}{24} \\
&\quad + \frac{\pi^2\ln^2 2}{36} - \frac{\ln^4 2}{36}\Big) + \Big\{\frac{41m^4}{96\lambda} + m^2\phi^2\Big(\frac{371}{96} - \frac{7}{4\sqrt{3}}\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big)\Big) \\
&\quad + \lambda\phi^4\Big(\frac{701}{384} - \frac{3\sqrt{3}}{4}\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big) + \frac{\zeta(3)}{4}\Big)\Big\}\ln\Big(\frac{m_\phi^2}{\bar{\mu}^2}\Big) - \Big\{\frac{17m^4}{48\lambda} + \frac{37m^2\phi^2}{24} \\
&\quad + \frac{143\lambda\phi^4}{192}\Big\}\ln^2\Big(\frac{m_\phi^2}{\bar{\mu}^2}\Big) + \Big\{\frac{5m^4}{48\lambda} + \frac{7m^2\phi^2}{24} + \frac{9\lambda\phi^4}{64}\Big\}\ln^3\Big(\frac{m_\phi^2}{\bar{\mu}^2}\Big)\Big]\,, \qquad (13)
\end{aligned}
$$

where m_ϕ^2 is defined as $m_\phi^2 \equiv m^2 + \frac{\lambda\phi^2}{2}$.

In summary, using the method of integration-by-parts recurrence relations, we have obtained the exact relations between the non-tetrahedron three-loop integrals $B_N(0,0,1,1,1,1)$, $D_5(1,1,1,1,1,0)$, and $D_5(1,1,1,1,2,0)$ and the tetrahedron three-loop integrals $B_N(1,1,1,1,1,1)$, $B_M(1,1,1,1,1,1)$, and $D_5(1,1,1,1,1,1)$, whose values are known to the finite parts. As an application of our loop calculations, the analytic evaluation of three-loop effective potential in the $\overline{\mathrm{MS}}$ scheme for the single-component massive ϕ^4 theory was obtained.

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